Making Mathematics Learning Meaningful and Responsive

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1. Introduction:

The purpose of this module is to develop an understanding of why mathematics is considered to be a difficult subject in school education. The module offers some working principles for planning and orchestrating mathematics classrooms which focus on developing conceptual understanding and promoting students' mathematical thinking. Further, the module focuses on activities that can be used to make mathematics meaningful and exciting for students, teachers and teacher educators. The suggested activities can be used by school leaders in different ways, such as, in continuous interactions or discussions with teachers, during workshops for mathematics teachers and students, for organizing *math-melas* in a school or a group of schools, as mathematics laboratory tasks, and so on.

2. Objectives

The module addresses the following objectives:

(a) Understanding students' difficulties with mathematics at the school level.

(b) Engaging with the reasons for students' difficulties and addressing them through conceptual activities.

(c) Building a deeper understanding of mathematics among School Heads, teachers and students.

(d) Learning to *discuss* mathematics, that is, read, write and talk about mathematics through engagement with different kinds of activities.

3. Concept Box

Conceptual Understanding, Knowticing, Math Tasks, Math talk, Meaningful Experiences, Responsiveness.

4. Organization of the Module

Mathematics is often considered to be a difficult subject at the school level. The popularly known reasons for this difficulty lie in the abstract nature of the subject, lack of connections between formal mathematics and other experiences within and outside school, emphasis on rote-memorization of formula, and difficulties in making mathematics enjoyable and meaningful for all learners. While it has been acknowledged that we need to build strong foundations at the primary level, the conceptual understanding in mathematics is an important aim for all levels of schooling.

This module is organized into five sections. The first section will involve a discussion on what makes mathematics difficult for school students. We will understand the reasons which make mathematics challenging and inaccessible for school students. In the second section, we will discuss some principles, from the existing literature on mathematics education, that can guide the teaching and learning of mathematics at all levels of schooling. These principles are oriented to making mathematics more meaningful for school students. We will discuss each of these principles using examples from school mathematics. The third section includes some suggestive activities that can be used for making mathematics meaningful and enjoyable for students. In the fourth section, we will discuss the support that mathematics teachers would need in promoting students' conceptual understanding of mathematics. In the last section, there will be some concrete suggestions for school leaders on enabling the school, teachers and community to become more responsive to students' learning of mathematics.

5. What Makes Mathematics Difficult?

In India, mathematics is a compulsory school subject. The continuous emphasis on mathematics from the early grades shows that it is important from the perspective of creating *educated* and *schooled* persons. As we begin to think about students' challenges with mathematics, we realize that several of these are guided by the popular public or societal perception about the nature of the subject and about students' abilities in mathematics.

5.1 Overemphasis on Performance

Figure 1: Students making nets of cube and cuboids

Students' "ability" in mathematics is largely determined by their performance in examination. The standardized examinations, at school and national level, as well as the expectations to participate in state and national level competitive exams pressurize students to perform at mathematics. From the very early stages of schooling, students begin to develop an understanding of themselves being either "good" or "bad" at mathematics. It becomes difficult for students to deviate from these frames since such thinking is often reinforced by their school and home environments. The societal perceptions about mathematics being a difficult subject further increases the helplessness of students who do not find themselves getting good grades in mathematics. Therefore, students who might be interested in the subject for different reasons and in different ways are left out because of performative imagination of mathematics at different stages of schooling.

Popular attitudes about mathematics and students' identities are merged together to exclude students with specific groups of students from doing mathematics. For example, blind students, girls, students from weaker socio-economic backgrounds are discouraged from pursuing mathematics in higher grades. The reasons vary from popular perceptions about the lack of ability among disadvantaged students to the increasing complexity of the subject later in the schooling. Such factors reinforce the discourse on students' lack of abilities rather than thinking about the way the subject is conceptualized and dealt within the educational spaces.

Knowledge of mathematics is often equated with general smartness and talent in our society. The discourse on talent and giftedness, because of higher performance in mathematics, excludes a majority of students who might be interested in pursuing mathematics. On one hand, some students are forced to do mathematics because of their high grades irrespective of their interest in other field such as, arts, music, physical education, history. These students are pushed and trained to participate in high stakes examination and represent the school, district, state, etc. in competitions. On the other, we label students who do not pursue mathematics as "less" talented. The labelling of students and their identification is being pushed to the stages of schooling and is often associated with the brand value of the school. Students who are labelled as failing in mathematics, referring to their performance, whose interest is not considered in deciding the choice of subjects that they might wish to pursue. The labelling of all kinds is problematic as it creates fixed trajectories for students. Such performance-oriented trajectories contradict the purpose of school education, which is to explore a variety of interests and make connections between different concepts, disciplines and ideas.

5.2 Nature of the Subject and the School Curriculum

Mathematics is considered to be an abstract subject and school learning is organized cumulatively over different grades. Students revisit concepts learnt in the previous grades with different levels of complexity and emphasis. While the recurrence of concepts at different levels can be seen as an opportunity to revisit conceptual understanding in different ways, it is often used with a fixed mindset. If a student does not "perform" well in Grade 9, it is assumed that they will not be good at mathematics at the later grades. Similarly, if a student finds division of fractions challenging, it is assumed that the student will face difficulties in learning decimal fractions and percentages. The accumulated nature of mathematics creates fixed mindsets in relation to students' ability and comfort with mathematics. Students are rarely given opportunities to revisit the topics in a conceptual manner. The abstractness and cumulative nature of mathematics are often used to reinforce that a majority of students are not capable of doing mathematics. In other words, the access to mathematics is limited to very few students by such popular perceptions. While both these perceptions about the nature of mathematics have some connection to the uniqueness of mathematics as a domain, understanding and dealing with them in a meaningful way is important.

Mathematics has a unique language with its own semantics and syntax. It uses symbols and statements which indicate different levels of claims. For example, a conjecture, mathematical statement, diagram, word problem, placement of numbers in a division algorithm - all of these are different ways in which mathematical language is used. While we all know about the difference between the language of mathematics and colloquial language, rarely explicit efforts are made to connect students' language with the mathematical language. For example, we are aware that students find it easier to solve a word problem in their own language as compared to the way they are presented in mathematics textbooks or examinations. We have several students who memorize how to write the proof of a theorem without engaging with the form and structure of proof writing. Each of these aspects of mathematical language need careful attention. Students' knowledge and language need to be used as a resource to connect and make mathematical language meaningful and accessible.

Mathematics curriculum is strictly framed within each level of the school education, that is, completion of multiplication of fractions before Grade 8 and introduction of set theory at Grade 10. Students are neither given adequate support nor are they encouraged to take less or more time in dealing with a specific topic. While we acknowledge that students have their unique ways of thinking and pace of learning, the organization of the school calendar, the expectations of the curriculum and frequency of assessments, does not allow interests to prosper in specific areas.

Making errors is a part of any natural process of learning. In mathematics, students' errors are interpreted as their lack of understanding or knowledge. There is a vast research literature on students' errors in different mathematical topics. This literature reveals different sources of students' errors, such as,

(a) overgeneralizations made from logical connections with prior knowledge, for example,
  (i) 6.73 is greater than 8.1 because 673 is greater than 81, and
  (ii) all the angles made in a polygon are less than 180° (One of the angle in concave polygon is more than 180^0).
(b) extending procedural explanations to make sense of newer situations, for example,
  (i) extending the explanation that when denominators are the same, just perform the required operation on the numerator, = $\frac{5}{7}-\frac{5}{3}=\frac{5}{4}$

(ii) students' explanations such as  is a triangle because it has three sides and three angles but  is not a triangle because the base is not straight!

Students' errors can be viewed as windows into students' thinking. Teachers might encourage students to explain their thinking when making an error, discuss this thinking with the whole class, encourage students in the class to offer mathematical explanations, through examples, counter examples, proofs, etc., which might help in addressing the root cause of erroneous responses.

The societal perceptions make mathematics challenging and difficult for students from diverse backgrounds. These challenges along with teaching practices which encourage rote memorization and discourage discussions and questioning in mathematics classrooms aggravate the problems in pursuing mathematics. In the next section, we will discuss some principles to guide mathematics teaching, which can help address the problems of mathematics learning. The task in Thinking Box 1 helps to understand the important of listening to students' thinking and plan ways of dealing with it.

Thinking Box 1

In 2018, Educational Initiatives published a series of videos on students' conceptions around different mathematical concepts. These videos show students from Grades 4, 6 and 8, from select Indian states, responding to some basic mathematical questions. Watch the video on students' conceptions about what is a triangle from the link below and think about the following questions.

Link: https://www.youtube.com/watch?v=g-Nda2tarcQ&t=308s

1. How are students defining a triangle?
2. Identify one similarity and one difference among different definitions given by the students.
3. What are the prototypical examples of triangles used in textbooks and in classroom teaching?

6. Principles for Teaching Mathematics

We will begin this section with some important insights from National Curriculum Framework 2005 (NCF 2005) and National Curriculum Framework for School Education 2023 (NCF 2023). These insights will be used to propose five principles for teaching mathematics. These principles are also derived from reflection on research with experienced school mathematics teachers:

1. Emphasizing mathematical processes and big ideas
2. *Knowticing* the mathematical potential of students' utterances, listening and responding in mathematically encouraging ways
3. Valuing conceptual reasoning along with procedural reasoning
4. Creating a culture of talking and discussing in mathematics classroom
5. Using a variety of explanations and examples

Each principle is explained through some examples from mathematical tasks from across levels of schooling (primary, secondary and senior secondary).

6.1 Vision for Mathematics Education in National Curriculum Framework

In India, the National Curriculum Frameworks (NCF) drafted by National Council of Educational Research and Training (NCERT), serve as the basis for design of syllabus and textbooks for all subjects in schooling. After India got independence, there have been five national curriculum frameworks, namely, The Curriculum for the Ten-year School: A Framework (NCF 1975), National Curriculum for Elementary and Secondary Education: A Framework (NCF 1988), National Curriculum Framework for School Education (NCF 2000), National Curriculum Framework (2005), National Curriculum Framework for School Education (NCF 2023). While these frameworks offer a range of ideas about the aims of school mathematics, approaches for teaching mathematics, organization of content at different levels, different states can adapt these frameworks based on the social and cultural context. Therefore, states often develop their own State Curriculum Frameworks (SCF) in their language. All these national and state curriculum frameworks are freely available for download on NCERT and SCERT websites.

All NCFs, in the section on mathematics, discuss the aims of mathematics, fear and failure of mathematics among students, organization of syllabus across grades, big ideas that need to be focused across different sub-domains (arithmetic, algebra, geometry, set theory, calculus), and suggestions for teachers and textbook writers. As teachers and school leaders, reading these national frameworks helps us understanding what the proposed aims of school education are and how these aims are linked with mathematics education. We will now discuss a few selected statements on mathematics learning at the school level from the two recent curriculum frameworks, NCF 2005 and NCF 2023.

Table 1: Mathematics Education from NCF 2023

1. Mathematics helps students develop not only basic arithmetic skills, but also the crucial capacities of logical reasoning, creative problem solving, and clear and precise communication (both oral and written). Mathematical knowledge also plays a crucial role in understanding concepts in other school subjects, such as Science and Social Science, and even Art, Physical Education, and Vocational Education. Learning Mathematics can also contribute to the development of capacities for making informed choices and decisions. Understanding numbers and quantitative arguments is necessary for effective and meaningful democratic and economic participation. (p. 269)
2. Mathematical knowledge is built through finding patterns, making conjectures (i.e., proposed truths), and then verifying/refuting those conjectures through logical and rigorous reasoning (i.e., through explanations/proofs or counter examples). The process of finding patterns, making conjectures, and finding proofs or counterexamples often involves a tremendous amount of creativity, sense of aesthetics, and elegance. Often, there are many different ways to arrive at the same mathematical truth and many different ways of solving the same problem. It is for that reason that mathematicians often refer to their own subject as more of an art than a science. (p. 270)
3. We must rethink the approach of teaching to one where students see Mathematics as a part of their life, and enjoy it with a greater focus on reasoning and creative problem solving... Students should not be allowed to fall behind in Mathematics and should be immediately supported to catch up if they do fall behind. NEP 2020 already has suggested delinking competitive entrance exams and the 'coaching culture' from the scheme of studies in schools. These measures should help redress this situation. (p. 272)

The three statements selected from NCF 2023, shown in Table 1, make suggestions for reimagining mathematics. The framework suggests that the need to make mathematics meaningful and enjoyable by suggesting a shift in the focus of mathematics teaching and learning. This shift is from rote memorization of formula and procedures to a focus on processes such as conjecturing, reasoning, finding and explaining patterns, formulating proofs and counter examples, etc. NCF 2023 also suggests that mathematics borrows from and connects with other school subjects such as arts, physical education and social sciences to create meaningful experiences for students. This integration is not limited to small school projects, although those could be useful starting points. Imagining the connections between subjects requires some careful planning, teaching and reflection to do justice to learning.

Table 2: Vision of School Mathematics (NCF 2005, p. 43)

1. Children learn to enjoy mathematics rather than fear it.
2. Children learn important mathematics: Mathematics is more than for formulas and mechanical procedures.
3. Children see mathematics as something to talk about, to communicate through, to discuss among themselves, to work together on.
4. Children pose and solve meaningful problems.
5. Children use abstractions to perceive relationships, to see structures, to reason out things, to argue the truth or falsity of statements.
6. Children understand the basic structure of mathematics: Arithmetic, algebra, geometry and trigonometry, the basic content areas of school mathematics, all offer a methodology for abstraction, structuration and generalisation.
7. Teachers engage every child in class with the conviction that everyone can learn mathematics.

Table 2 presents the vision of school mathematics from the NCF 2005. NCF 2005 laid the ground for distinguishing between the higher and narrow aims of mathematics education at the school level. The narrow aims of mathematics learning include useful capabilities of computation, solving problems on different topics with a particular focus on arithmetic of numbers, fractions and percentages. On the other hand, developing students' ability to think mathematically, to conjecture and understand ways of solving problems, offer mathematical reasons and pose problems constitute the higher aims of learning mathematics. In order to

Thinking Box 2

National Curriculum Frameworks, NCF 2005 and NCF 2023, suggest an emphasis on processes in learning mathematics. Some of these processes include problem posing, estimation and approximation, conjecturing and proving, formulating mathematics statements, representing, visualizing, etc.

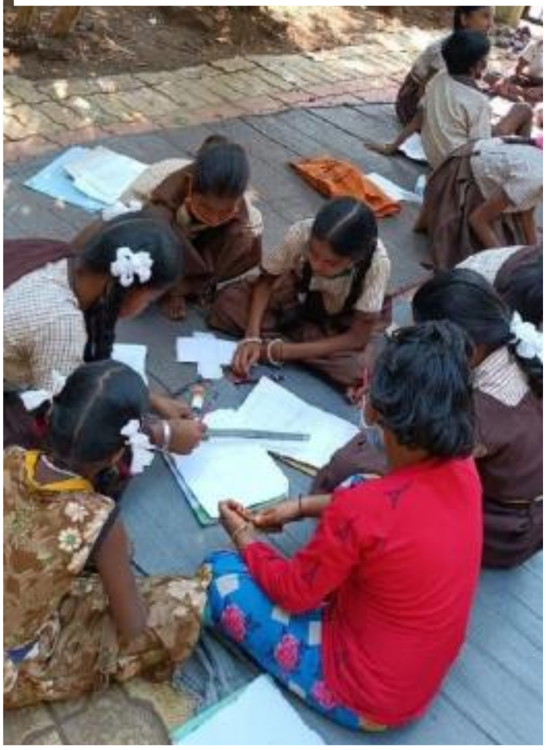
1. Discuss, with mathematics teachers, how these processes can be focused in the teaching of mathematics by taking specific examples of mathematical topics from primary and secondary level.

pursue the higher aim, curriculum, teaching and assessment methods would require rethinking. Thinking Box 2 poses questions that can be discussed among school teachers and leaders to concretely plan how mathematical processes can be emphasized in school teaching and learning.

How can mathematics be made meaningful and relevant to students? Can the abstract mathematics be made enjoyable for students? How can the structure and language of mathematics be made accessible for diverse students? Can fun and learning go together in mathematics classrooms? These are important questions for us to deliberate so that students can be engaged with mathematics meaningfully and without the fear of failure. Communicating mathematically is an important skill that school mathematics can encourage and inculcate. Eliciting students' questions, partial or incorrect responses, asking students to explain their thinking and give reasons in words can contribute to making mathematics something to talk about. In the next section, we will discuss some activities to encourage students to learn by engaging with mathematics actively.

6.2 Guiding Principles

Fig 2: Students work on creating cubes



The aim of making mathematics meaningful are providing students with the tools of mathematic learning can be strived by changing our approach towards mathematics teaching and learning. Some principles, which are derived from research on experienced mathematics teachers, that can guide teachers and teacher educators are discussed below. Each of these are exemplified using a mathematics problem. The readers are suggested to think of more examples for each principle and how these can be suitably modified and used in their work.

An important part of teachers' work is to make connections between content and students' thinking. Students think in different ways and their thinking gets manifested in their written and oral work through their struggles, questions, explanations, etc. Understanding students requires not just planning but also some in-the-moment thinking and responding by teachers. For example, many a times, during teaching, students ask a question that the teacher might have not planned. Similarly, when a student does not understand an explanation given by another student or the teacher, it is expected that the teacher bridges the gap between students' current understanding and the expected explanation. *Knowticing* refers to a combination of knowledge and noticing. While a deeper knowledge of the content requires that teachers understand the depth and breadth of the content, anticipating students' possible ways of thinking and reasoning is an important part of this work. Noticing means an awareness of and responsiveness to students' utterances. Teachers see beyond what the students speak, that is, they "see" the mathematical potential underlying a students' response and can use it as an opportunity for discussions in classroom. Thinking Box 3 suggests a task where teachers are encouraged to examine a student's conjecture and discuss it mathematically.

Thinking Box 3

Think about a mathematical observation or question that a student might have asked you or your colleague. Some examples of students' questions are listed below. You may select these if you prefer.

Some Examples of students' observations

1. $\frac{9}{7}$ is greater than $\frac{9}{5}$ because 9 is the same and 7 is greater than 5.
2. The two lines CD and EF are perpendicular because you can see that.
3. By SSA, the two triangles are congruent.
4. The sum of two odd numbers is an even number. I know because 7 + 3 = 10.
5. When the perimeter increases, area also increases for a polygon.

After selecting a student's observation or question, think and discuss the following questions.

1. Rephrase the students' observation in different words.
2. What is the reason for students' observation?
3. How is the reason connected to the teaching of that topic?
4. Which mathematical *process,* such as, representation, conjecturing, proving

Tables 3 and 4 have excerpts from two classrooms, where the teacher is introducing the place value names of the decimal fraction. In both these classrooms, a student asks the question about, the position of oneths place value. Can you identify the differences in the way that the two teachers deal with the students' observation? How do these change the course of classroom discussion? The two excerpts are different in the way that the teachers respond to the question. The teacher in the second classroom *knowtices* the mathematical potential of the student's question and then uses it to steer the classroom discussions towards understanding the reasons for the non-existence of this place value.

Excerpt 1	
Teacher	Like we have tens, we have tenths. Like hundreds there is hundredths. So for thousand we will have?
Students	Thousandths
Teacher	The difference is that in decimals (place value), we use "ths". Can you repeat all the decimal place values?
Students	Tenths, Hundredths, Thousandths
Student	Teacher, where is oneths?
Teacher	Oneths?
Student	Yes teacher, tens tenths hundreds hundredths ones oneths.
Teacher	Oneths does not exist. Okay now we will put the numbers in the place value table.

Excerpt 2	
Teacher	See here we have decimal place values; we use "ths" in them. Like see tenths, hundredths, thousandths. These names are similar to the whole number place value. What is the place value of digits in whole number?
Students	Teacher tens, hundreds, thousands, lakhs.
Teacher	Yes, like that we have decimal place value. But the places are different. What are they?
Students	Tenths, Hundredths, Thousandths
Student R	Teacher, why oneths is not there?
Teacher	Hmm okay, so Rishi is asking why these is no oneths. What do you (all) think?
Teacher	Like tens and tenths, after the (decimal) point, there is no oneths, why?

Student D	Teacher, I know, when we do one part of one, will be one.
Student A	No ma'am, there cannot be a oneths place in the decimal part.
Teacher	Why?
Student D	Ma'am, because one is a whole number and tenths means starting with ones, this whole number [one] has ten parts.

It requires a deeper knowledge of the content to unpack the mathematical potential of the students' questions. It is not that the teacher in Excerpt 2 knew the answer to the oneths question. However, the teacher did not dismiss the student's question and decided to discuss it with the whole class. She listened to students' explanations - which are conceptually sound and justify the absence of the oneths place value. Listening to students' responses, which are often partial, incorrect, less coherence, or stated in their mother tongue, carefully is important. Teachers also need to think about different ways of responding to students' ideas - in ways that encourage students to participate actively in mathematics classroom. This participation needs to come from mathematical engagement -classroom norms that can encourage students to question each other's and the teacher's explanations, offer explanations, and challenge them. Students are often discouraged from asking questions, giving explanations, and solving problems both independently and with partners. It is important that students' observations (partial, incorrect, different) are encouraged. In Excerpt 1, the explanation given by the teacher, "oneths does not exist" is not a mathematical explanation. The students are expected to believe the teacher. We have to shift the focus to mathematical explanations, where students learn to evaluate their statements using mathematical tools.

Many a times, we (teachers, school leaders, parents) think - how can students solve a problem unless taught? Often students use their prior knowledge to make sense of the problem and then make attempts at solving it. Wherever possible, students need to be given such opportunities to decode the problem, identify and recall what they know, and attempt solutions as they work with each other. We notice that in Excerpt 2, all the students' explanations are conceptual in nature. They are building two different kinds of explanations -(a) first if tenths is defined as $\frac{1}{10}$ then oneths would be defined as $\frac{1}{1}$ -which is the same as the ones position, (b) second, one tenths is $\frac{1}{10}$ times ones, so there is no missing place value in between ones and tenths. The teacher used both of these explanations to justify why oneths does not exist. However, this process of re-voicing the students' question, posing it to the whole class, encouraging students to offer and build on explanations, helped the teacher in formulating these conceptual explanations. Students' knowledge is a reliable source for teachers to work with in the classroom. Another source is research on specific topics and on students' mathematical thinking. Students may also refer to materials such as texts, discuss with their immediate and other colleagues, and become a part of different groups or communities for upgrading their knowledge, and so on. Before offering conceptual explanations to the students, it is helpful that teachers ask themselves for reasons for mathematical procedures, formulae and algorithms. Learning to justify the procedure is an important part of learning mathematics. Why do we carry over in the addition of numbers, why is negative times a negative number positive, why do we decide to choose digits in the division algorithm, why is the volume of a cylinder calculated through $\pi r^2 h$? Lastly, a focus on big ideas in Mathematics might help teachers to place emphasis on processes of mathematics than the pieces of content. Big ideas include those mathematical ideas which run across different concepts and are central to learning mathematics (One that links numerous mathematical understanding into a coherent whole). Instead of procedures or actions, they align with the mathematical meanings. For example, consider the big idea on *equivalence*. Equivalence runs across arithmetic and algebra. Students encounter this idea in the school mathematics curriculum in variety of ways, such as, (a) writing equivalent numerical and algebraic statements, (b) balancing and solving numeric and algebraic equations, (c) properties of arithmetic, like, commutativity and inverse, (d) creating equivalent fractions, (e) creating equivalent forms in equation solving, etc. Table 5 presents some examples of big ideas in mathematics.

Table 5: Big Ideas in mathematics (Charles & Carmel, 2005)

1. The set of real numbers is infinite and each real number can be associated with a unique point on the number line.
2. Base ten numeration system uses ten digits 0 to 9, and their place value is determined by their position in the number.
3. Any number, measure, numerical and algebraic expression, algebraic equation can be represented in different ways that have the same value.
4. Numbers, expressions and measures can be compared using their relative values.
5. The same number sentence can be associated with different real life situations, and different number sentences can be associated with the same real life situation.

6. For a given set of numbers, there are some relationships which are always true - these relationships are referred as properties. These relationships are justified using properties of arithmetic and algebra.
7. Basic facts and algorithms use the notions of equivalence to transform calculations into simpler ones.
8. Numbers and measures can be approximated by replacing them with closest measures which are easy to compute and are familiar referents.
9. Relationships can be described, and generalizations can be made from repeated patterns. However, to know that a relationship is always true, it needs to be proved.
10. Mathematical situations and structures can be represented abstractly using variables, expressions and equations.
11. If two quantities vary proportionally, their relationship can be represented as a linear function.
12. Mathematical relationships and data can be described using words, tables, graphs, and equations.
13. The notion of equivalence helps in solving numerical and algebraic equations by transforming equations and inequalities into perceptually different but equivalent forms.
14. Two- and three-dimensional objects with or without curved surfaces can be described, analysed and classified based on their attributes.
15. Objects in space can be oriented and transformed in infinite ways and these ways can be described and analysed mathematically.
16. The chance of an event occurring can be described numerically by a number between 0 and 1 inclusive. The chance is used to make predictions about events.

Thinking Box 4 suggests tasks where teachers and school leaders may identify explicitly where and how big ideas underlie the teaching of school mathematics.

Thinking Box 4

1. Align the big ideas with different topics and sub-topics from different grades. Give examples.
2. What does it mean to develop the "mathematical" culture in your classroom? Discuss with your math teachers and share at least five ways in which this culture can be constructed with the help of students.
3. Discuss with teachers about ways in which teachers' *knowticing* can be strengthened using a study of students' answer sheets for exams?

7. Suggested Tasks and Activities

In this section, we will discuss five activities including tasks for students. We will use the term *mathematical activity* to make the expectations of the task explicit, that is, reflect on how students will engage with the task. We will then take examples of activities and determine tasks for students within each. While the tasks roughly correspond to different levels of schooling, they can be extended to other levels of schooling. Some of these extensions will be discussed. The principles discussed in Section 6.2 may be recalled in thinking about, planning and engaging with these activities in classroom teaching. The same activities can be used by School Heads with teachers for reflection and deepening their knowledge.

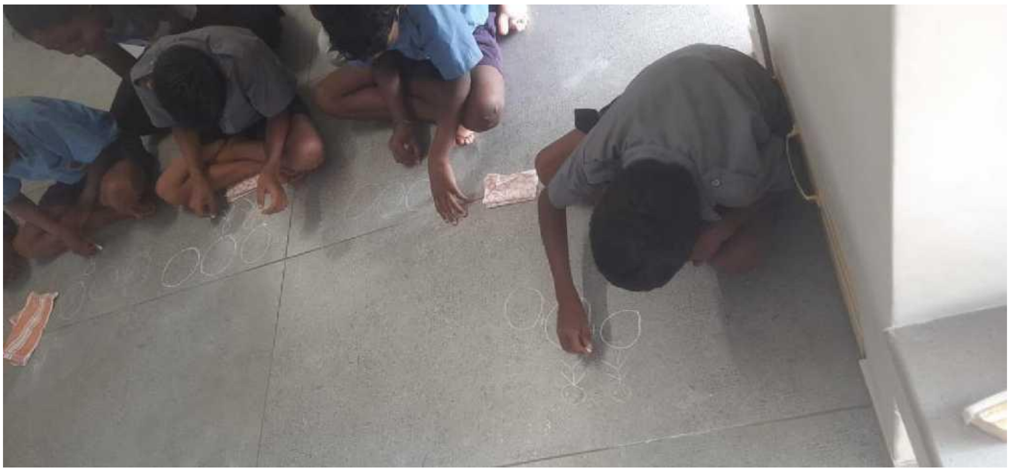


Figure 3: Students trying to show 3 rotis divided among 2 people

7.1 Activity 1 - Routine task conceptual focus

The first activity involves doing routine tasks but with a conceptual focus. Table 6 shows two different tasks. In the first task, students are asked to subtract two numbers but using three different representations. The second task requires students to evaluate the truth of a generalized statement.

Table 6: Routine task, conceptual focus

1. Subtract 53 from 71 in three different ways. Try to use different representations in the three ways.
2. *Subtraction always makes the number smaller, and multiplication always makes the number bigger*. Do you agree with the above statement? Give reasons for your answer.

How can a subtraction problem be represented in different ways? It can be shown using taking away of bundles and sticks, through breaking up numbers using place value (70 and 1 minus 50 and 3), breaking up numbers to get friendly numbers (71 minus 51 and then minus 2), showing them on a number line and taking jumps backwards, using the subtraction algorithm of borrowing, etc. While each of these approaches uses a different representation, diagrammatic, symbolic manipulation, etc.; it would be helpful to take a step further and ask students to find connections between any of two these approaches. For example, what is the connection between using the number line, the friendly numbers, and the algorithm -how are these methods similar and different from each other. Such questions help students find conceptual meanings underlying procedures, explore different ways of representing the same problem, and seek connections between different ways of problem solving.

The mathematical generalization made in the second task is a question raised by many students. In primary mathematics, students use phrases such as, more or bigger which are associated with addition and multiplication. Sometimes teachers also think that these generalizations are infact true. To examine the two parts of the statement - we can either prove them to be true or find at least one counterexample to say that these are false. When we subtract a smaller whole number from a bigger whole number, the result is always smaller than the bigger whole numbers. It may or may not be smaller than the subtrahend, for example, compare, 72 – 50 and 72 – 30. But what if we change the order of the numbers or we subtract integers. How does that change the result of subtraction? We would, therefore, need to examine different cases of subtraction. Consider the four examples given below.

(a) 53 – 44 (b) 44 – 53 (c) – 44 – 53 (d) 44 – (–53)

All of these can be called as subtraction problems. Encouraging students to give these examples would be useful in identifying different cases when making generalizations. As we solve each of these four examples, we see that, in (d) the subtraction does not make the number smaller. Some might argue that the operation is not that of subtraction, but addition here. However, we need to distinguish between the operation of subtraction, and the negative sign of the number. Similarly, can you find different cases for multiplication and then identify the counter examples? The hint is to consider rational number multiplication.

7.2 Activity 2 - Meaningful tasks

While several data collection exercises are done by students, they are rarely familiarized with the messiness of the actual data, ways of cleaning up, the process of categorization, and making interpretations. In this activity, students can be given actual data from the newspapers, such as, poll data, investment or temperature data. Alternatively, students may be asked to collect some actual data, such as, census, hours of study for working children, expenditure of high-, middle- and lower-income families, influence on local temperature due to climate change, how a digital application measures steps and represents data, etc.

Table 7: Task on data

Situation: We will find out about the expenditure of families who belong to different socio-economic groups. Broadly, there are three income groups: lower, middle and higher. There are also other groups such as upper middle class and lower middle class. But for this assignment, we will use the three broad categories. The assignment can be done in pairs or triads (groups).	
1.	Design five questions that will help you understand the income of a family.
2.	Discuss these questions with your friends and teachers. Revise them if needed.
3.	Request time with each of the three families - one from each category. Collect data from these three families.
4.	Do you need to revise your questions based on the response from the three families? Do you want to frame the questions differently for each group?
5.	After revising the questions, collect data from 30 families - 10 from each category.
6.	How will you organize this data?
7.	After organization, what are the patterns that are emerging?
8.	Which findings would you like to discuss with your classmates and why?
9.	You want to draw everyone's attention to the key findings that you have selected. Which mode of data representation (line graph, bar graph, pie chart, scatter plot, etc.) would you choose for this data and why?
10.	Prepare a presentation using sheets - one or two sheets for each stage of data collection. Share your methods and findings.

The task in activity 2 is not just about collecting data for also preparing questions for data collection, organizing and cleaning data, identifying outliers, making interpretations from data, and choosing an appropriate tool for data representation.

7.3 Activity 3 - Relevance and connections

Mathematical knowledge has to be relevant and meaningful. Also, the need for connecting mathematics with real-life situations is well established. What becomes important in this persistent call is identifying and using situations through which students understand the need for learning mathematical ideas. The third activity encourages students to use materials from their surroundings and ask mathematical questions - much closer to the need for emergence of the topic. This task is on measurement, but similar tasks can be designed within each sub-domain of mathematics and for different levels of schooling.

Take any four water containers of different shapes and sizes from your surroundings. Note that these containers do not have measures marked on them.

Table 8: Task on measurement

a.	How do you know which container has the least capacity?
b.	Tell a friend on the phone about the relation between capacities of all the four containers?
c.	When telling a friend, did you try to find a container which both of you have? What is the process of standardization of measures? Why do we need it?
d.	What is the ratio of the capacity of the container with largest capacity and a one-liter bottle?
e.	For which questions did you use the estimate measures, why?
f.	For which questions did you find the actual capacity of the container, why?

Data can be presented to students in many forms. For example, score board from different cricket matches can be used to understand concepts of decimal fractions, average, speed, etc. Similarly, graphs used to show rising and falling temperature or poll results can be used for reading and interpretation. It would be helpful to discuss with student questions such as, what kind of assumptions are made in such data collection processes, how the data is collected and cleaned, what kind of mathematics is useful in collecting and working with large data sets, and so on.

Thinking Box 5

1. Each activity mentioned in Section 7 creates opportunities for using some guiding principles and some big ideas. Discuss with your math teachers, which big ideas and guiding principle does each activity has the potential of addressing.
2. Take any one activity and try to extend it to teach other big ideas.

8. Supporting mathematics teachers: Suggestions for Leaders

Mathematics teachers need continuous and meaningful support for their learning and reflection on practice. In this section, we will discuss some ways of supporting teachers which have been prevalent in in-service teacher professional development. School leaders and teacher educators would need to provide and modify this support in the long term as well through continuous interactions with teachers. While workshops might be helpful, the interactions remain limited in their scope and depth. Supporting teachers on a regular basis might help create a community where teachers can be encouraged to talk about and reflect on their teaching, learn from their students and each other.

8.1 Workshop tasks on problems of practice

Workshops are the most common mode of professional development in India. However, many a times, the workshops deal with ideas which are far from teachers' lived realities. The objective and content of the workshop needs to be guided by some conceptual and practical principles. For example, it is important to discuss with teachers about procedural and conceptual explanations, and big Ideas in mathematics as these have direct relevant to what and how they teach mathematics. Similarly, discussions around students who struggle with mathematics, how to work towards changing attitudes towards students who can and who cannot do mathematics, encouraging students to participate actively in a mathematics classroom - all can be culturally structured in a teacher education or teacher learning space. In these workshops, teachers may also be familiarized with research on students and topics, which would support their planning and justifications. Different kinds of material resources can be brought in for discussion and reflection. Similarly, the workshop space can be organized in a way that the group develops into a community of learning, relying on their and each other's experiences and knowledge. Teachers may also be encouraged to participate actively in deciding the topics of these workshops.

8.2 Supporting teachers in classroom

School leaders and teacher educators would need to extend the spaces of learning from workshop settings to classrooms. Classrooms offer a rich variety of opportunities for learning for teachers and supporting teachers in the contexts of their practice would be helpful in supporting reformed pedagogies. There are various ways of supporting teachers in classrooms. Traditionally, teacher educators and school leaders observe teachers in the classrooms for the purpose of inspection or evaluation. Supporting teachers would require that instead of judging teachers, such opportunities of observations are used for discussion and reflection on students' thinking, classroom culture, mathematical explanations, equitable participation, and conceptual learning. School Leader or Teacher educators may co-teach or plan a lesson with the teachers which experiments with specific pedagogies or explanations. They may teach a lesson and invite all mathematics teachers to observe and reflect on the lesson. It is also useful to support the teachers while they are teaching by supporting individual students whom teacher may need help with.

8.3 Creating a professional learning community

Teacher professional development can be envisioned towards forming learning communities with teacher and teacher educators who meet constantly to discuss the problems of Practice, such opportunities may be provided by the School Leaders in their respective schools. Activities in the learning community may involve invitation to teachers to share their experiences and learn from each other, examining research on students' thinking and learning to identify their usefulness for practice, collectively designing classroom experiments and sharing insights about their learning, deliberating on the challenges of practice, and working towards their own professional development. Teachers can collectively design and analyse materials that are used for practice. For example, designing worksheets for students, creating and using geoboards for teaching, formulating and testing activities, etc.

Thinking Box 6

How can professional learning communities be utilised for

(a) Planning content to be taught in classrooms

(b) Organizing math melas in school

(c) Designing assessments where students work on projects in groups

(d) Collecting and analysing data from classroom

9. Summary

In this module, we have discussed how it is important to understand the reasons for fear and failure in mathematics; and address attitudes which discourage students from pursuing mathematics. We learn from national curricula that students need to find mathematics meaningful and engaging. This requires that the teacher education space exemplifies productive mathematical practices, such as, focus on big ideas in mathematics, encouraging questioning in mathematics classroom, finding reasons and explanations for mathematical procedures, etc. Some suggestions are made on how school leaders can aim to create professional learning communities which are practice centered and provide the necessary support structure to the teachers. These suggestions are made through thinking boxes, tasks and activities, and section on supporting teachers.

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11. Assessment

The following are multiple choice questions. Some questions have more than one correct answer. After selecting the correct answer(s), discuss with a colleague why your answer is correct.

1.	Why is it important to understand the reasons for students' difficulties with mathematics?	
	(a) Offer windows into students' thinking	(b) Address students' mistakes
	(c) Evaluate students accordingly	(d) All of the above
2.	How can teachers enhance their *knowticing* of mathematics underlying students' partial responses?	
	(a) Discussing students' responses and reasons	(b) Giving marks or grades to students
	(c) Reflecting on teaching decisions	(d) Discouraging students from giving partial statements
3.	What are some of the key features of activities that make them meaningful and relevant?	
	(a) Students' mathematical participation	(b) Linking activity with the mathematical concept
	(c) Fun and enjoyment without conceptual connection	(d) Encouraging students to provide different solutions
4.	Which of these are NOT big ideas in learning mathematics?	
	(a) Making mathematical relationships explicit	(b) Discussing the underlying mathematical structure
	(c) Connecting estimation and exact measures	(d) Using identities to solve algebraic equations

5.	What is the difference between problem posing and problem solving?	
	(a) They are the same	(b) Problem posing encourages students to understand the organization of problems
	(c) Problem solving is more important	(d) Problem posing is more important
6.	What are the ways in which mathematics teachers can be supported in the contexts of their practice?	
	(a) Workshops outside classrooms	(b) Co-teaching or supporting them with students
	(c) Reflecting on students' work	(d) Reflecting on teaching trajectories and curriculum